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ON LINEAR ALGEBRAIC SEMIGROUPS

BY

MOHAN S. PUTCHA

ABSTRACT. Let K be an algebraically closed field. By an algebraic semigroup we mean a Zariski closed subset of K^n along with a polynomially defined associative operation. Let S be an algebraic semigroup. We show that S has ideals $I_0, \ldots, I_t$ such that $S = I_t \supseteq \cdots \supseteq I_0$, I_0 is the completely simple kernel of S and each Rees factor semigroup I_k/I_{k-1} is either nil or completely 0-simple ($k = 1, \ldots, t$). We say that S is connected if the underlying set is irreducible. We prove the following theorems (among others) for a connected algebraic semigroup S with idempotent set $E(S)$. (1) If $E(S)$ is a subsemigroup, then S is a semilattice of nil extensions of rectangular groups. (2) If all the subgroups of S are abelian and if for all $a \in S$, there exists $e \in E(S)$ such that $ea = ae = a$, then S is a semilattice of nil extensions of completely simple semigroups. (3) If all subgroups of S are abelian and if S is regular, then S is a subdirect product of completely simple and completely 0-simple semigroups. (4) S has only trivial subgroups if and only if S is a nil extension of a rectangular band.

1. Preliminaries. Throughout this paper, $\mathbf{Z}^+$ will denote the set of all positive integers. If X is a set, then $|X|$ denotes the cardinality of X. K will denote a fixed algebraically closed field. If $n \in \mathbf{Z}^+$, then $K^n = K \times \cdots \times K$ is the affine n-space and $\mathfrak{M}_n(K)$ the algebra of all $n \times n$ matrices. If $A \in \mathfrak{M}_n(K)$, then $\rho(A)$ is the rank of A. In this paper we only consider closed sets with respect to the Zariski topology. So $X \subseteq K^n$ is *closed* if and only if it is the set of zeroes of a finite set of polynomials on K^n. Let $X \subseteq K^m$, $Y \subseteq K^n$ be closed, $\varphi: X \to Y$. If $\varphi = (\varphi_1, \ldots, \varphi_m)$ where each φ_i is a polynomial, then φ is a *morphism*. Let $p, n \in \mathbf{Z}^+$, $p \leqslant n$. Then we use, without further comment, the well-known fact that the set $T = \{A \mid A \in \mathfrak{M}_n(K), \rho(A) < p\}$ is closed. In fact for $A \in \mathfrak{M}_n(K)$, $A \in T$ if and only if all minors of A of order $\geqslant p$ vanish. By an *algebraic semigroup* we mean $(S, \circ)$ where $\circ$ is an associative operation on S, S is a closed subset of K^n for some $n \in \mathbf{Z}^+$ and the map $(x, y) \to x \circ y$ is a morphism from $S \times S$ into S. If S has an identity element then S is an *algebraic monoid*. Polynomially defined associative operations on a field have been studied by Yoshida [21], [22], Plemmons and Yoshida [13]. Yoshida's results have been generalized to integral domains by Petrich [12]. Clark [4] has studied semigroups of matrices forming a linear variety. Algebraic monoids are briefly encountered in Demazure and Gabriel [8]. The author [15] has studied semigroups on affine spaces defined by polynomials of degree at most 2.

Let S be an arbitrary semigroup. If S has an identity element, then $S^1 = S$. Otherwise $S^1 = S \cup \{1\}$, $1 \notin S$, with obvious multiplication. If $a \in S$, then the

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centralizer of a in S, $C_S(a) = \{x | x \in S,\ xa = ax\}$. Then *center* of S, $C(S) = \bigcap_{a \in S} C_S(a)$. If $a, b \in S$, then $a|b$ (a *divides* b) if $b \in S^1aS^1$. $\mathcal{J}$, $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$ will denote the usual Green's relations on S (see [6]). If $a \in S$, then we let $J(a) = S^1aS^1$. J_a, H_a will denote the $\mathcal{J}$-class and $\mathcal{H}$-class of a in S, respectively. $E(S)$ will denote the set of idempotents of S. If $e, f \in E(S)$ then $e < f$ if $ef = fe = e$. An idempotent semigroup is called a *band*. A commutative band is called a *semilattice*. A band satisfying the identity $xyzw = xzyw$ is called a *normal band*. If $ab = b$ [$ba = b$] for all $a, b \in S$, then S is a *right* [*left*] *zero semigroup*. A direct product of a right zero semigroup and a left zero semigroup is a *rectangular band*. A direct product of a rectangular band and a group is a *rectangular group*. A direct product of right [left] zero semigroup and a group is a *right* [*left*] *group*. Let I be an ideal of S. If S/I is a nil semigroup, then we say that S is a *nil extension* of I. Let δ_α ($\alpha \in \Gamma$) be a set of congruences on S. If $\bigcap_{\alpha \in \Gamma} \delta_\alpha$ is the equality congruence, then S is a *subdirect product* of S_α ($\alpha \in \Gamma$). See [7, p. 99] for details.

A congruence δ on S is an $\mathbb{S}$*-congruence* if S/δ is a semilattice. If S is a disjoint union of subsemigroups S_α ($\alpha \in \Gamma$) and if for each $\alpha, \beta \in \Gamma$, there exists $\gamma \in \Gamma$ such that $S_\alpha S_\beta \cup S_\beta S_\alpha \subseteq S_\gamma$, we will say that S is a *semilattice* (union) of S_α ($\alpha \in \Gamma$). We also say that S_α ($\alpha \in \Gamma$) is a *semilattice decomposition* of S. There is an obvious natural correspondence between $\mathbb{S}$-congruences and semilattice decompositions [6, p. 25]. A semigroup with no $\mathbb{S}$-congruences other than $S \times S$ is said to be $\mathbb{S}$*-indecomposable*. Let ξ be the finest $\mathbb{S}$-congruence on S. Throughout this paper, we let $\Omega = \Omega(S) = S/\xi$ denote the maximal semilattice image of S. By a theorem of Tamura [18], [19], each component of ξ is $\mathbb{S}$-indecomposable and is called the $\mathbb{S}$*-indecomposable component* of S. An ideal P of S is *prime* if $S \setminus P$ is a subsemigroup of S. In such a case $\{P, S \setminus P\}$ is a semilattice decomposition of S. If S is a commutative algebraic semigroup, then it follows from Corollary 1.4 below, and Tamura and Kimura [20] that $E(S) = \Omega(S)$.

Let S, T be algebraic semigroups, φ: $S \to T$ a (semigroup) homomorphism. Then φ is a **-homomorphism* if φ is also a morphism (of varieties). If φ is a bijection and if both φ and φ^{-1} are *-homomorphisms, then we say that φ is a **-isomorphism* and that S, T are **-isomorphic*. S is *connected* if the underlying closed set is irreducible (i.e. is not a union of two proper closed subsets).

The proof of the following result can be found in Demazure and Gabriel [8, II, §2, Theorem 3.3].

THEOREM 1.1 (SEE [8]). *Let S be an algebraic monoid. Then S is *-isomorphic to a closed submonoid of $\mathfrak{M}_n(K)$ for some $n \in \mathbf{Z}^+$.*

Let S be an algebraic semigroup which is a group. Let φ: $S \to \mathfrak{M}_n(K)$ be given by Theorem 1.1. By Hilbert's Nullstellensatz, $1/\det \varphi(x)$ is a polynomial on S. So $x^{-1} = \varphi^{-1}(\operatorname{adj} \varphi(x)/\det \varphi(x))$. Hence the map $x \to x^{-1}$ is a morphism and S is an algebraic group in the usual sense [2].

The following result can also be found in [8, II, §2, Corollary 3.6].

COROLLARY 1.2 (SEE [8]). *Let S be an algebraic monoid which is not a group. Then the nonunits of S form a closed prime ideal of S.*

Let S be an algebraic semigroup, $S \subseteq K^n$. Let $T = (S \times \{0\}) \cup \{(0, 1)\} \subseteq K^{n+1}$. Define $(x, \alpha)(y, \beta) = (xy + \beta x + \alpha y, \alpha\beta)$. Then T is an algebraic monoid with identity element $(0, 1)$. S is *-isomorphic to the closed subsemigroup $S \times \{0\}$ of T. Hence we have

COROLLARY 1.3. *Let S be an algebraic semigroup. Then S is *-isomorphic to a closed subsemigroup $\mathcal{M}_n(K)$ for some $n \in \mathbf{Z}^+$.*

The following result was pointed out to the author by Clark [5].

COROLLARY 1.4 [CLARK]. *Let S be an algebraic semigroup. Then there exists $n \in \mathbf{Z}^+$ such that for all $a \in S$, a^n lies in a subgroup of S.*

PROOF. By Corollary 1.3, we can assume that S is a closed subsemigroup of $\mathcal{M}_n(K)$ for some $n \in \mathbf{Z}^+$. Let $A \in S$. Then without loss of generality we can assume $A = \left(\begin{smallmatrix} B & 0 \\ 0 & C \end{smallmatrix}\right)$. where $B \in \mathcal{M}_p(K)$ is invertible and $C \in M_{n-p}(K)$ is nilpotent. Let $T = \{\left(\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right) | X \in M_p(K), X \text{ is invertible}\}$. Then $A^n \in T$. Let $G = \{(\left(\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right), \alpha) | X \in M_p(K), \alpha \in K, \alpha \det \cdot X = 1\}$. Then G is an algebraic group, $\varphi\colon G \to T$ given by $\varphi((\left(\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right), \alpha) = X$ is a bijective morphism. $G_1 = \varphi^{-1}(T \cap S)$ is a closed subsemigroup of G_1. It is well known (see [8, II, §2, Corollary 3.5]) that a closed submonoid (and hence a closed subsemigroup) of a linear algebraic group is a subgroup. Thus G_1 is a subgroup of G. So $T \cap S = \varphi(G_1)$ is a subgroup of S. Since $A^n \in T \cap S$, we are done. $\square$

COROLLARY 1.5. *Let S be an algebraic semigroup. Then S has a kernel M which is closed and completely simple.*

PROOF. By Theorem 1.1, we can assume that S is a closed subsemigroup of $\mathcal{M}_n(K)$ for some $n \in \mathbf{Z}^+$. By Corollary 1.4 and Clark [3], S has a completely simple kernel M given by its elements of minimal rank r. Then $M = \{a | a \in S, \rho(a) < r + 1\}$ is closed. $\square$

LEMMA 1.6. *Let $\mathcal{E}$ be an infinite set of idempotents in $\mathcal{M}_n(K)$ of rank r. Then there exist $E, F \in \mathcal{E}$ such that $E \neq F$ and $\rho(EF) = \rho(FE) = r$.*

PROOF. If $E \in \mathcal{E}$, then let $\mathcal{A}_E = \{A | A \in \mathcal{M}_n(K), \rho(EA) < r\}$, $\mathcal{B}_E = \{A | A \in \mathcal{M}_n(K), \rho(AE) < r\}$. Then $\mathcal{A}_E$, $\mathcal{B}_E$ are closed subsets of $\mathcal{M}_n(K)$. We claim:

$$\begin{array}{l}\text{There exists an infinite subset } \mathcal{F} \text{ of } \mathcal{E} \\ \text{such that for all } E \in \mathcal{F}, |\mathcal{A}_E \cap \mathcal{F}| < \infty.\end{array} \tag{1}$$

Suppose (1) is false. Then there exists $E_1 \in \mathcal{E}$ such that $\mathcal{E}_1 = \mathcal{A}_{E_1} \cap \mathcal{E}$ is infinite. Again by (1), there exists $E_2 \in \mathcal{E}_1$ such that $\mathcal{E}_2 = \mathcal{A}_{E_2} \cap \mathcal{E}_1$ is infinite. Continuing, we obtain a sequence $E_1, E_2, \ldots,$ in $\mathcal{E}$ such that $\rho(E_iE_j) < r$ for $i < j$. So $E_{i+1} \in \mathcal{A}_{E_1} \cap \cdots \cap \mathcal{A}_{E_i}$, $E_{i+1} \notin \mathcal{A}_{E_{i+1}}$. Hence

$$\mathcal{A}_{E_1} \supsetneqq \mathcal{A}_{E_1} \cap \mathcal{A}_{E_2} \supsetneqq \mathcal{A}_{E_1} \cap \mathcal{A}_{E_2} \cap \mathcal{A}_{E_2} \supsetneqq \cdots.$$

Since $\mathcal{A}_{E_i}$'s are closed sets, we have a contradiction to the Hilbert Basis Theorem. Thus (1) is true. The dual of (1) applied to $\mathcal{F}$ shows that there exists an infinite subset $\mathcal{G}$ of $\mathcal{F}$ such that for all $E \in \mathcal{G}$, $|\mathcal{B}_E \cap \mathcal{G}| < \infty$. Let $E \in \mathcal{G}$. Then

$|\mathcal{A}_E \cap \mathcal{G}| < \infty$, $|\mathcal{B}_E \cap \mathcal{G}| < \infty$. Hence there exists $F \in \mathcal{G}$ such that $F \notin \mathcal{A}_E \cup \mathcal{B}_E$. So $\rho(EF) = \rho(FE) = r$. $\square$

A semigroup S with the property that a power of each element lies in a subgroup of S is said to be *strongly π-regular*. The study of strongly π-regular rings and semigroups was initiated by Azumaya **[1]**, Drazin **[9]** and Munn **[11]**. Clark **[3]** showed that a strongly π-regular matrix semigroup has a kernel given by its elements of minimal rank. Let S be a strongly π-regular semigroup. A $\mathcal{J}$-class of S containing an idempotent is called *regular*.

THEOREM 1.7. *Let S be a strongly π-regular subsemigroup of $\mathfrak{M}_n(K)$. Then S has only finitely many regular $\mathcal{J}$-classes.*

PROOF. Suppose not. Then there exists an infinite set of idempotents $\mathcal{E}$ of S such that for all $e, f \in \mathcal{E}$, $e \mathcal{J} f$ implies $e = f$. Let $r = 0, \ldots, n$, let $\mathcal{E}_r = \{e | e \in \mathcal{E}, \rho(e) = r\}$. Then $\mathcal{E}_r$ is infinite for some r. By Lemma 1.6, there exist $e, f \in \mathcal{E}_r$ such that $e \neq f$, $\rho(ef) = \rho(fe) = r$. Let $\mathcal{V}$ be the space of all $n \times 1$ vectors on K. Then $ef\mathcal{V} = e\mathcal{V}$, $fe\mathcal{V} = f\mathcal{V}$. Hence $e\mathcal{V} = (ef)^t\mathcal{V}$ for all $t \in \mathbf{Z}^+$. There exists an idempotent g of S, $p \in \mathbf{Z}^+$ such that $g\mathcal{H}(ef)^p$. Hence $g\mathcal{V} = (ef)^p\mathcal{V} = e\mathcal{V}$. If $v \in \mathcal{V}$, then $ev \in g\mathcal{V}$ and so $gev = ev$. So $ge = e$. Hence $f|(ef)^p|g|e$. So $f|e$. Similarly $e|f$ and $e\mathcal{J}f$. This contradiction proves the theorem. $\square$

COROLLARY 1.8. *Let S be a strongly π-regular subsemigroup of $\mathfrak{M}_n(K)$. Then $\Omega(S)$ is finite.*

PROOF. Let φ: $S \to \Omega(S)$ denote the natural homomorphism. By Theorem 1.7, $\varphi(E(S))$ is finite. Let $a \in S$. Then $a^n\mathcal{H}e$ for some $e \in E(S)$. So $\varphi(a) = \varphi(a^n) = \varphi(e)$. Hence $\Omega(S) = \varphi(S) = \varphi(E(S))$ is finite. $\square$

LEMMA 1.9. *Let S be a strongly π-regular semigroup with only finitely many regular $\mathcal{J}$-classes. Then there exist finitely many ideals $I_0, \ldots, I_t$ of S such that $S = I_t \supseteq \cdots \supseteq I_0$, I_0 is the completely simple kernel of S and each I_i/I_{i-1} is either completely 0-simple or a nil semigroup ($i = 1, \ldots, t$).*

PROOF. We prove by induction on the number of regular $\mathcal{J}$-classes of S. Let $E = E(S)$. Let $J_{e_1}, \ldots, J_{e_n}$ be the regular $\mathcal{J}$-classes of S where $e_1, \ldots, e_n \in E$. Let $I = J(e_1) \cap \cdots \cap J(e_n)$. Then I is an ideal of S. So there exists $f \in I \cap E$. Let $a \in S$. Then there exists $m \in \mathbf{Z}^+$ such that $a^m \mathcal{J} e_i$ for some i. So $f \in J(a^m) \subseteq J(a)$. Hence $J(f) = I_0$ is the kernel of S. By Munn **[11]**, I_0 is completely simple. Let $\mathcal{K} = \{J(e) | e \in E \cap (S \setminus I_0)\}$. Then $\mathcal{K}$ is finite. If $\mathcal{K} = \varnothing$, then $S \setminus I_0$ has no idempotent and $S \setminus I_0$ is nil. So assume $\mathcal{K} \neq \varnothing$. Then $\mathcal{K}$ has a minimal element $J(g)$, $g \in E$. Let $I_2 = J(g)$, $I_1 = I_2 \setminus J_g$. Then $I_0 \subseteq I_1$ and I_1 is an ideal of S. Let $a \in I_1$. Then $a^m \mathcal{J} h$ for some $h \in E$, $m \in \mathbf{Z}^+$. Then $h \in I_1$. So $J(h) \subsetneqq J(g)$. By minimality of $J(g)$, $h \in I_0$. Thus $a^m \in I_0$. So I_1/I_0 is nil. Since $I_2 \setminus I_1 = J_g$ and $g \in E$, I_2/I_1 is 0-simple. By Munn **[11]**, I_2/I_1 is completely 0-simple. Clearly S/I_2 has lesser number of regular $\mathcal{J}$-classes than S. We are thus done by our induction hypothesis. $\square$

By Theorem 1.7 and Lemma 1.9 we have the following.

THEOREM 1.10. *Let S be a strongly π-regular subsemigroup of $\mathfrak{M}_n(K)$. Then there exist ideals $I_0, \ldots, I_t$ of S such that $S = I_t \supseteq \cdots \supseteq I_0$, I_0 is the completely simple kernel of S and each I_i/I_{i-1} is either completely 0-simple or nil ($i = 1, \ldots, t$).*

By Corollary 1.3, Corollary 1.4 and Theorem 1.10, we have

COROLLARY 1.11. *Let S be an algebraic semigroup. Then S has ideals $I_0, \ldots, I_t$ such that $S = I_t \supseteq \cdots \supseteq I_0$, I_0 is the completely simple kernel of S and each I_i/I_{i-1} is either completely 0-simple or nil ($i = 1, \ldots, t$).*

THEOREM 1.12. *Let S be an algebraic semigroup and P a prime ideal of S. Then P is closed.*

PROOF. By Corollaries 1.3 and 1.4, we can assume that S is a closed, strongly π-regular subsemigroup of $\mathfrak{M}_n(K)$ for some $n \in \mathbf{Z}^+$. Hence $S_1 = S \setminus P$ is strongly π-regular. By Clark [3] the kernel T of S_1 is the set of elements of S_1 of minimal rank. Let $e \in E(T)$, $\rho(e) = r$. Let $a \in S_1$. Then $(eae)^n \in T$ and so $\rho((eae)^n) = r$. Let $a \in P$. There exists $f \in E(P)$ such that $(eae)^n \mathcal{H} f$. So $ef = fe = f$. Hence $\rho(f) < \rho(e) = r$. Clearly $\rho((eae)^n) = \rho(f)$. Thus $P = \{a | a \in S,\ \rho((eae)^n) < r\}$ is closed. $\square$

2. Connected algebraic semigroups. Let S be an algebraic semigroup, $e \in E(S)$. Then the maximal subgroup H_e of S need not be closed. However H_e can be identified with $G = \{(a, b) | a, b \in S,\ ab = ba = e,\ ae = ea = a,\ be = eb = b\}$. If $(a, b), (c, d) \in G$, define $(a, b)(c, d) = (ac, db)$. Then G is an algebraic group. The correspondence between H_e and G is given by $a \leftrightarrow (a, a^{-1})$. More precisely define $\varphi: G \to S$ as $\varphi(a, b) = a$. Then φ is an injective *-homomorphism and $\varphi(G) = H_e$. It is easy to show that G is unique to within *-isomorphisms. It can also be easily shown that if S is connected then so is G. However, we will not need these facts in this paper.

THEOREM 2.1. *Let S be a connected algebraic semigroup. Then $\Omega(S)$ has an identity element.*

PROOF. Let $\Omega = \Omega(S)$, $\varphi: S \to \Omega$ be the canonical homomorphism. By Corollary 1.8, Ω is a finite semilattice. Suppose Ω has two maximal elements e, f. Then $\Omega_1 = \Omega \setminus \{e\}$, $\Omega_2 = \Omega \setminus \{f\}$ are prime ideals of Ω, $\Omega = \Omega_1 \cup \Omega_2$. So $S = P_1 \cup P_2$ where $P_i = \varphi^{-1}(\Omega_i)$, $i = 1, 2$. But P_1, P_2 are prime ideals of S and hence closed by Theorem 1.12. This contradiction shows that Ω has a maximum element e. So e is the identity element of Ω. $\square$

In the above notation, we call $\varphi^{-1}(e)$ the *top $\mathcal{S}$-indecomposable component* of S. If S is a monoid, then the top $\mathcal{S}$-indecomposable component of S is the group of units of S.

PROPOSITION 2.2 *Let S be a connected algebraic semigroup, $e, f \in E(S)$. Then eS, Se, eSf are connected, closed subsemigroups of S. If SeS is closed, then SeS is also connected.*

PROOF. $eS = \{x | x \in S, ex = x\}$, $eSf = \{x | x \in S, ex = x = xf\}$. Hence eS, Se, eSf are closed. Define φ_1: $S \to eS$ as $\varphi_1(x) = ex$. Since φ_1 is a surjective morphism, eS is connected. Define φ_2: $S \to eSf$ as $\varphi_2(x) = exf$. Since φ_2 is a surjective morphism, eSf is connected. $S \times S$ is connected. Define φ_3: $S \times S \to SeS$ as $\varphi_3(x, y) = xey$. If SeS is closed, then $\varphi_3(S \times S) = SeS$ is also connected. □

THEOREM 2.3. *Let S be a connected algebraic semigroup. Then*

(1) *all maximal subgroups of S are closed if and only if S is a nil extension of a completely simple semigroup.*

(2) *all subgroups of S are trivial if and only if S is a nil extension of a rectangular band.*

PROOF. (2) follows trivially from (1). So we prove (1). First assume that all maximal subgroups of S are closed. Let $e \in E(S)$. By Proposition 2.2, eSe is connected. By hypothesis H_e is closed. By Corollary 1.2, $eSe \setminus H_e$ is also closed. Hence $eSe = H_e$. Thus $a|e$ for all $a \in S$. Hence $e \in T =$ kernel of S. Thus $E(S) \subseteq T$. By Corollary 1.11, T is completely simple and S/T is nil. Conversely assume S/T is nil where T is the completely simple kernel of S. Then for $e \in E(S) = E(T)$, $H_e = eSe$ is closed. □

THEOREM 2.4. *Let S be a connected algebraic semigroup. Then the following conditions are equivalent.*

(1) *All subgroups of the top $\mathbb{S}$-indecomposable component of S are abelian.*

(2) *All subgroups of S are abelian.*

(3) *eSe is commutative for all $e \in E(S)$.*

PROOF. (1) ⇒ (3). Let T be the top $\mathbb{S}$-indecomposable component of S. Then by Theorem 1.12, $P = S \setminus T$ is closed. Let $e \in E(T)$. Then H_e is abelian. Let $S_1 = eSe$, $P_1 = S_1 \setminus H_e$. Then P_1 is closed, S_1 is closed and connected. $S_1 = P_1 \cup H_e$. Let $a \in H_e$. Then $H_e \subseteq C_{S_1}(a)$ and so $S_1 = P_1 \cup C_{S_1}(a)$. Hence $C_{S_1}(a) = S_1$. Thus $H_e \subseteq C(S_1)$ and $S_1 = P_1 \cup C(S_1)$. Hence $C(S_1) = S_1$ and S_1 is commutative. Let $a \in T$. By Corollary 1.4 there exists $n \in \mathbf{Z}^+$ such that $a^n \mathcal{H} e$ for some $e \in E(T)$. So $a^nSa^n \subseteq eSe$ is commutative. Let $T_1 = \{a | a \in S, a^nSa^n$ is commutative$\}$. Then T_1 is closed, $T \subseteq T_1$. Since $S = P \cup T_1$, $T_1 = S$. Hence eSe is commutative for all $e \in E(S)$. That (3) ⇒ (2) ⇒ (1) is obvious. □

THEOREM 2.5. *Let S be a connected algebraic semigroup such that all subgroups of S are abelian. Suppose further that for each $a \in S$, there exists $e \in E(S)$ such that $ea = ae = a$. Then S is a semilattice of nil extensions of completely simple semigroups and the top $\mathbb{S}$-indecomposable component of S is completely simple.*

PROOF. By Theorem 2.4, eSe is commutative for all $e \in E(S)$. Let $a \in S$. Then there exists $e \in E(S)$ such that $ea = ae = a$. Let $x, y \in S^1$. Then $xay\, xay = x(eae)(eyxe)(eae)y = x(eae)^2(eyxe)y = xa^2yxey$. Hence $a^2|(xay)^2$. By a paper by the author [14, Theorem 2.13], S is a semilattice of nil extensions of completely simple semigroups. Let T be the top $\mathbb{S}$-indecomposable component of S. Then T is a nil extension of a completely simple semigroup. Let $T_1 =$ kernel of T. Then

$E(T) \subseteq T_1$. Let $a \in T$. Then there exists $e \in E(S)$ such that $ea = a$. Clearly $e \in E(T)$. Hence $a \in T_1$ and $T = T_1$ is completely simple. $\square$

A semigroup is *regular* if $a \in aSa$ for all $a \in S$.

THEOREM 2.6. *Let S be a regular, connected algebraic semigroup such that all subgroups of S are abelian. Then S is a finite subdirect product of semigroups, each of which is either completely simple or completely* 0-*simple.*

PROOF. Let $e, f, g \in E(S)$ such that $e \geqslant f$, $e \geqslant g$, $f \mathcal{J} g$. We claim that $f = g$. There exist $x, y \in S^1$ such that $xfy = g$. By Theorem 2.4, eSe is commutative. So $g = exefeye = efeeyeexe = eyeexeefe$. Hence $g = fyexe = eyexf$. Hence $gf = fg = g$. So $f \geqslant g$. Similarly $g \geqslant f$ and $f = g$. Thus

$$\begin{gathered}\text{for all } e, f, g \in E(S), \text{ the conditions}\\ e \geqslant f, e \geqslant g, f \mathcal{J} g \text{ together imply } f = g.\end{gathered} \tag{2}$$

By Munn [**11**], the principal factors of S are completely simple or completely 0-simple. Hence by (2) and Lallement [**10**, Theorem 2.17], S is a subdirect product of completely simple and completely 0-simple semigroups. Since, by Theorem 1.7, S has only finitely many $\mathcal{J}$-classes, a close examination of [**10**] shows that the subdirect product can be chosen to be finite. $\square$

THEOREM 2.7. *Let S be a connected algebraic semigroup, $e \in E(S)$, $e \notin C(S)$. Then $e \in B$ where B is either an infinite, closed right zero subsemigroup of S or an infinite, closed left zero subsemigroup of S.*

PROOF. By symmetry assume $\dim eS \geqslant \dim Se$. Define ϕ: $eS \to eSe$ as $\varphi(a) = ae$. Then φ is a surjective morphism, eS, Se, eSe are closed and connected. By [**17**, Chapter I, §6, Theorem 7], $\dim eS \geqslant \dim eSe$. First assume $\dim eS = \dim eSe$. Then since $eSe \subseteq eS$, $eS = eSe$. Since $\dim eS \geqslant \dim Se$ and $eSe \subseteq Se$ we have $Se = eSe$. Hence $eS = Se$ and $e \in C(S)$, a contradiction. So $\dim eS > \dim eSe$. Let $B = \varphi^{-1}(e)$. By [**17**, p. 60] $\dim B > 0$. Hence B is infinite. Let $a \in B$. Then $a \in eS$, $ae = e$. Hence $a = ea$. Let $a, b \in B$. Then $ab = aeb = eb = b$. This proves the theorem. $\square$

THEOREM 2.8. *Let S be a connected algebraic semigroup. Then the following conditions are equivalent.*

(1) *$E(S)$ is finite.*

(2) *$E(S)$ is commutative.*

(3) *$E(S) \subseteq C(S)$.*

(4) *S is a semilattice of nil extensions of groups.*

PROOF. By Theorem 2.7, (1) $\Rightarrow$ (3) and (2) $\Leftrightarrow$ (3). By Corollary 1.8, (4) $\Rightarrow$ (1). So it suffices to show that (3) $\Rightarrow$ (4). Assume (3). Let $a, b \in S$ such that $ab, ba \in E(S) \subseteq C(S)$. Then $ab = a(ba)b = (ab)(ba) = baba = ba$. By Weissglass and the author [**16**, Corollary 8], we are done. $\square$

The following result is implicit in Munn [**11**]. We include a proof here for the convenience of the reader.

LEMMA 2.9 [MUNN]. *Let S be a strongly π-regular semigroup. Let $a, b \in S$. If $a\mathcal{J}ab$ then $a\mathcal{R}ab$. If $a\mathcal{J}ba$, then $a\mathcal{L}ba$. If $a\mathcal{J}a^2$, then $a\mathcal{H}a^2$.*

PROOF. It suffices to consider the case $a\mathcal{J}ab$. There exist $x, y \in S^1$ such that $xaby = a$. Then $x^t a(by)^t = a$ for all $t \in \mathbf{Z}^+$. There exist $n \in \mathbf{Z}^+$, $e \in E(S)$ such that $(by)^n \mathcal{H} e$. So $a = ae \in a(by)^n S^1 \subseteq abS^1$. Hence $a\mathcal{R}ab$. □

LEMMA 2.10. *Let S be a connected algebraic semigroup, $e, f \in E(S)$, $e|f$. Then there exists $g \in E(S)$ such that $e\mathcal{R}g$ and $gf\mathcal{L}f$.*

PROOF. Let $E = E(S)$. Suppose the lemma is false. Then by Lemma 2.9, $gf \nmid f$ for all $g \in E$ with $g\mathcal{R}e$. In particular $ef \nmid f$. There exist $x, y \in S$ such that $xey = f$. By Corollary 1.2 and Proposition 2.2, $eSe \setminus H_e$ and $fSf \setminus H_f$ are closed sets. Let

$$T_1 = \{a | a \in eS, fxaf \in fSf \setminus H_f\}, \qquad T_2 = \{a | a \in eS, ae \in eSe \setminus H_e\}.$$

Then T_1, T_2 are closed subsets of eS. If $e \notin T_1$, then $fxef \in H_f$ and $ef|f$, a contradiction. So $e \in T_1$. Clearly $fxeyf = f$ and so $ey \notin T_1$. Thus $\varnothing \neq T_1 \subsetneq eS$. Clearly $e \notin T_2$. We claim that $ef \in T_2$. Otherwise $efe \in H_e$. Then $ef|efe|e|f$, a contradiction. So $ef \in T_2$. Hence $\varnothing \neq T_2 \subsetneq eS$. Since eS is connected by Proposition 2.2, $T_1 \cup T_2 \neq eS$. Hence there exists $a \in eS$ such that $a \notin T_1 \cup T_2$. Then $ea = a$, $fxaf \in H_f$, $ae \in H_e$. There exists $z \in S$ such that $zae = e$. So $za^2 = zaea = ea = a$. Hence $a^2\mathcal{J}a$. By Lemma 2.9, $a^2\mathcal{H}a$. By [**6**, Theorem 2.16], there exists $g \in E$ such that $a\mathcal{H}g$. Now $g \in a^2S = aeaS \subseteq aeS = eS$, $e \in aeS \subseteq aS = gS$. So $e\mathcal{R}g$. Now $fxagf = fxaf \in H_f$. Hence $gf|f$, a contradiction. This proves the lemma. □

THEOREM 2.11. *Let S be a connected algebraic semigroup such that $E(S)$ is a subsemigroup of S. Then S is a semilattice of nil extensions of rectangular groups.*

PROOF. Let $E = E(S)$. Let $a, b \in S$ such that $e = ab$, $f = ba \in E$. By the author [**14**, Theorem 2.17], it suffices to show that $fef = f$. Now $e = ab|(ba)^2 = f$. By Lemma 2.10 there exists $g \in E$ such that $e\mathcal{R}g$, $gf\mathcal{L}f$. Since $gf \in E$, $fgf = f$. Since $e\mathcal{R}g$, $eg = g$. So $fegf = f$. Since $fe \in E$, $fef = (fe)^2gf = fegf = f$. This proves the theorem. □

THEOREM 2.12. *Let S be a connected algebraic semigroup such that all subgroups of S are abelian. Then the following conditions are equivalent.*

(1) $E(S)$ is a band.

(2) $E(S)$ is a normal band.

(3) S is a semilattice of nil extensions of rectangular groups.

PROOF. (1) ⇒ (3) follows from Theorem 2.11. (2) ⇒ (1) is obvious. So we must show (3) ⇒ (2). By Corollary 1.4, there exists $n \in \mathbf{Z}^+$ such that for all $a \in S$, $a^n\mathcal{H}e$ for some $e \in E(S)$. Let $E = E(S)$ and let T be the top $\mathcal{S}$-indecomposable component of S. If $T = S$, we are done. So assume $P = S \setminus T \neq \varnothing$. P is a prime ideal of S and hence closed by Theorem 1.12. T is a nil extension of a rectangular group T_1. Since the subgroups of T_1 are abelian, T_1 satisfies the identity $xyzw = xzyw$. Hence T satisfies the identity $x^ny^nz^nw^n = x^nz^ny^nw^n$. By [17, p. 54], $S \times S \times S \times S$ is connected. Let $M = \{(a, b, c, d) | a, b, c, d \in S,\ a^nb^nc^nd^n = a^nc^nb^nd^n\}$. Then M is closed and $T \times T \times T \times T \times \subseteq M$. Clearly

$$\begin{aligned} S \times S \times S \times S = M &\cup (S \times S \times S \times P) \cup (S \times S \times P \times S) \\ &\cup (S \times P \times S \times S) \cup (P \times S \times S \times S). \end{aligned}$$

Hence $M = S \times S \times S \times S$. Thus for all $e, f, g, h \in E(S)$, $efgh = egfh$. In particular $efef = eeff = ef$ and $E(S)$ is a normal band. $\square$

THEOREM 2.13. *Let S be a connected algebraic semigroup such that* $\dim S = 1$. *Then S is either a group, a group with zero, a null semigroup, a right zero semigroup or a left zero semigroup.*

PROOF. First assume S has an identity element 1. If $E(S) = \{1\}$, then S is a group. Otherwise there exists $e \in E(S)$ such that $e \neq 1$. Then $\dim eS = \dim Se = 0$. So $eS = Se = \{e\}$. Hence S has a zero 0 and $E(S) = \{1, 0\}$. Let G be the group of units of S. By Corollary 1.2, $M = S \setminus G$ is a closed ideal of S. Let $a \in M$. Consider the map φ: $S \to M$ given by $\varphi(x) = ax$. φ is a morphism. Hence T, the closure of $\varphi(S)$ is irreducible. Since $T \subseteq M \neq S$, $\dim T = 0$. Since $0, a \in T$, $a = 0$. Thus $S = G \cup \{0\}$.

So assume S does not have an identity element. Let $e \in E(S)$. Suppose $eS = S$. Then $Se \neq S$. So $Se = \{e\}$. Let $a, b \in S$. Then $ab = a(eb) = (ae)b = eb = b$. So S is a right zero semigroup. Similarly $Se = S$ implies that S is a left zero semigroup. So assume $eS \neq S$, $Se \neq S$ for all $e \in E(S)$. Hence $eS = Se = \{e\}$. So S has a zero 0 and $E(S) = \{0\}$. By Corollary 1.4, there exists $n \in \mathbf{Z}^+$ such that $a^n = 0$ for all $a \in S$. Let $D = \{a | a \in S,\ a^2 = 0\}$. Then D is closed. Define φ: $S \to D$ as $\varphi(a) = a^{n-1}$. Then $\varphi(S) \neq \{0\}$. Let T be the closure of $\varphi(S)$. Then $T \subseteq D$. Since S is connected, T is irreducible. So $\dim T = 1$ and $S = T = D$. Let $a \in S$. Let $M = \{b | b \in S,\ ab = 0\}$. We claim that $M = S$. Suppose not. Clearly M is closed. Define ψ: $S \to M$ as $\psi(b) = ab$. Since $M \neq S$, $\psi(S) \neq \{0\}$. If W is the closure of $\psi(S)$, then $\dim W \neq 0$, W is irreducible, $W \subseteq M$. This contradiction shows that $M = S$. Hence $S^2 = \{0\}$, proving the theorem. $\square$

REMARK 2.14. It is well known [2, p. 257] that a connected algebraic group of dimension one is *-isomorphic to either $(K, +)$ or the group $\{(a, b) | a, b \in K,\ ab = 1\}$ under multiplication. Let S be an algebraic semigroup of dimension 1. The only case of Theorem 2.13 that needs a closer look is when $S = G^0$, G is a group. Let 1 be the identity of S. Then $\hat{G} = \{(a, b) | a, b \in S,\ ab = 1\}$ is a connected algebraic group of dimension 1. So S is isomorphic to $(K, \cdot)$. The example $S = \{(x, y) | x, y \in K,\ x^2 = y^3\}$ under multiplicaton shows that in general S is not *-isomorphic to $(K, \cdot)$.

THEOREM 2.15. *Let S be a connected algebraic semigroup such that* $\dim S = 2$. *Then $E(S)$ is a band. If S does not have an identity element then $E(S)$ is a normal band.*

PROOF. Let M be the kernel of S. By Corollary 1.5, M is closed and completely simple. Let $e \in E(M)$. Then $SeS = M$. By Proposition 2.2, M is connected. First assume $M = S$. If $eS = S$ for some $e \in E(S)$, it follows (since S is completely simple) that $E(S)$ is a right zero semigroup. Similarly $Se = S$ implies $E(S)$ is a left zero semigroup. So assume $eS \neq S$, $Se \neq S$ for all $e \in E(S)$. So $\dim eS = \dim Se \leqslant 1$ for all $e \in E(S)$. Let $e \in E(S)$. If $eSe \neq \{e\}$ then $\dim eSe = 1$. Since $eSe \subseteq Se \cap eS$, we obtain $eS = Se$. But then S is a group. So assume $eSe = \{e\}$ for all $e \in E(S)$. Then S is a rectangular band.

Next assume $\dim M = 1$. By Theorem 2.13, M is either a right zero semigroup, a left zero semigroup or a group. By symmetry assume M is not a left zero semigroup. If $E(M) = E(S)$, we are done. So assume $E(M) \neq E(S)$. Suppose S has an identity element 1. Let $e \in E(S)$, $e \notin M$. Then $M \subsetneqq eS$. So $eS = S$ and $e = 1$. Then $E(S) = E(M) \cup \{1\}$ and we are done. Next assume S does not have an identity element. Let $e \in E(S) \setminus M$. As above, $eS = S$. So $Se \neq S$. Now Me is closed and connected and $Me \subsetneqq Se$. So $\dim Me = 0$. If $Me = \{f\}$, let $\theta(e) = f \in E(M)$. So $\theta\colon E(S) \setminus M \to E(M)$. Let $D_1 = E(M), D_2 = E(S) \setminus D_1$. Then D_1, D_2 are right zero semigroups. If $e \in D_2$, $f \in D_1$, then $ef = f$, $fe = \theta(e)$. It follows easily that $E(S)$ is a normal band.

Finally assume that $\dim M = 0$. Then S has a zero 0. Suppose S has an identity element 1. Let $e \in E(S)$, $e \neq 1, 0$. Then $\{0\} \subsetneqq eSe \subseteq eS \subsetneqq S$. So $eS = eSe$. Similarly $Se = eSe$ and $e \in C(S)$. Hence $E(S) \subseteq C(S)$. Next assume S does not have an identity element. By symmetry we can assume that $eS \neq S$ for all $e \in E(S)$. Then $\{0\} \subsetneqq eSe \subseteq eS \subsetneqq S$ for all $e \in E(S)$, $e \neq 0$. So $eSe = eS$ for all $e \in E(S)$, $e \neq 0$. Let $A = \{e | e \in E(S),\ Se = s\}$. Then $A = \varnothing$ or A is a left zero semigroup. Let $e \in E(S)$, $e \neq 0$, $e \notin A$. Then $\{0\} \subsetneqq eSe \subseteq Se \subsetneqq S$. Hence $eSe = Se$ and $eS = Se$. So $e \in C(S)$. It follows that $E(S)$ is a normal band. □

Let S be a strongly π-regular semigroup, J a regular $\mathcal{J}$-class of S. Let J^0 be the semigroup $J \cup \{0\}$ where 0 is the zero of J^0 and for $a, b \in J$, we set $ab = 0$ if $ab \notin J$. By Munn **[11]**, J^0 is completely 0-simple. By the Rees theorem **[6**, Theorem 3.5] we can assume that $J^0 = (\Gamma \times G \times \Lambda) \cup \{0\}$ with sandwich map $P\colon \Lambda \times \Gamma \to G^0$ where G is a group. Multiplication in J^0 is given by

$$(\alpha, a, \beta)(\gamma, b, \delta) = \begin{cases} (\alpha, aP(\beta, \gamma)b, \delta) & \text{if } P(\beta, \gamma) \neq 0, \\ 0 & \text{if } P(\beta, \gamma) = 0. \end{cases} \tag{3}$$

THEOREM 2.16. *Let S be a connected algebraic semigroup, J a regular $\mathcal{J}$-class of S. Let J^0 have the Rees representation given by* (3). *Then for all $\alpha, \beta \in \Gamma$, there exists $\gamma \in \Lambda$ such that $P(\gamma, \alpha) \neq 0$ and $P(\gamma, \beta) \neq 0$. For all $\gamma, \delta \in \Lambda$, there exists $\alpha \in \Gamma$ such that $P(\gamma, \alpha) \neq 0$ and $P(\delta, \alpha) \neq 0$.*

PROOF. The second statement being the dual of the first, we only need to prove the first. Let $\alpha, \beta \in \Gamma$. Since J^0 is regular, it follows **[6**, Lemma 3.1] that there exist $\mu, \nu \in \Lambda$ such that $P(\mu, \alpha) \neq 0$, $P(\nu, \beta) \neq 0$. Let $e = (\alpha, P(\mu, \alpha)^{-1}, \mu)$, $f = (\beta, P(\nu, \beta)^{-1}, \nu)$. Then $e, f \in E(S)$, $e | f$. By Lemma 2.10, there exists $g \in E(S)$ such that $e \mathcal{R} g$ and $gf \mathcal{L} f$. Now $g = (\alpha, a, \gamma)$ for some $a \in G$, $\gamma \in \Gamma$. Since $g^2 = g$, $P(\gamma, \alpha) \neq 0$. Since $gf \neq 0$ in J^0, $P(\gamma, \beta) \neq 0$. This proves the theorem. □

THEOREM 2.17. *Suppose S is a connected, algebraic semigroup. Assume that S is a semilattice of groups and that $E(S)$ is linearly ordered. Then $|E(S)| \leqslant 2$.*

PROOF. By Theorem 2.8, $E(S)$ is finite and $E(S) \subseteq C(S)$. Suppose $|E(S)| \geqslant 3$. Let $E(S) = \{e_1 < e_2 < e_3 < \cdots\}$. Let $T = e_3 S$. Let $T_1 = e_1 S$, $T_2 = e_2 S$. Then $T_1 \subsetneqq T_2 \subsetneqq T$, $e_2(T \setminus T_1) = T_2 \setminus T_1$, $e_2 T_1 = T_1$. Define φ: $T \to T_2$, as $\varphi(x) = e_2 x$. Clearly φ is surjective and $\dim T > \dim T_2$. So [**17**, p. 60], $\dim \varphi^{-1}(a) > 0$ for all $a \in T_2$. In particular $\dim \varphi^{-1}(e_1) > 0$. Let $x \in \varphi^{-1}(e_1)$. Then $e_2 x = e_1$. But then $x \in T_1$ and so $e_2 x = x$. This contradiction proves the theorem. $\square$

3. Examples and problems. Let D be a closed subset of K^n. Let $\circ$ be a binary operation on D such that the map $(a, b) \to a \circ b$ from $D \times D$ into D is a morphism. We will then say that $(D, \circ)$ is an *algebraic groupoid.*

EXAMPLE 3.1. Let D be an algebraic groupoid, S a subsemigroup of D. Let T be the closure of S in D. Then T is an algebraic semigroup. In fact let $a \in S$, $T_1 = \{b | b \in T,\ ab \in T\}$. Then $S \subseteq T_1$ and so $T_1 = T$. So a $T \subseteq T$. Let $T_2 = \{b | b \in T,\ bT \subseteq T\}$. $S \subseteq T_2$ and so $T_2 = T$. Hence $T^2 \subseteq T$. Let $a, b \in S$ and let $T_3 = \{c | c \in T,\ (ab)c = a(bc)\}$. $S \subseteq T_3$ and so $T_3 = T$. Repeating this argument twice, we see that T is a semigroup.

EXAMPLE 3.2. Let $X \subseteq K^n$ be closed. Let $S = \{A | A \in \mathfrak{M}_n(K),\ XA \subseteq X\}$. Then S is a closed subsemigroup of $\mathfrak{M}_n(K)$.

EXAMPLE 3.3. Let $X \subseteq K^n$ be a nonempty closed set. Then X admits a right zero, left zero and null semigroup structures given by $ab = b$, $ab = a$, $ab = u$ where u is a fixed element of X.

EXAMPLE 3.4. Let S be any finite semigroup. Then S is closed subsemigroup of the finite dimensional algebra $K[S]$. Hence S is an algebraic semigroup.

EXAMPLE 3.5. Let $S \subseteq K^2$ be the closed set $\{(a, b) | a, b \in K,\ ab^2 = b\}$. If (a, b), $(c, d) \in S$, define $(a, b)(c, d) = (abcdac + 1 - abcd, 0)$. Then S is a commutative algebraic semigroup. Note that $(1, 1)S = S^2 = \{(a, 0) | a \in K, a \neq 0\}$ is not closed. $S^3 = \{(1, 0)\}$.

PROBLEM 3.6. Let S be an algebraic semigroup. Does there exist $n \in \mathbf{Z}^+$ such that $S^n = S^{n+1}$ is closed?

PROBLEM 3.7. Let S be an algebraic semigroup, $e \in E(S)$. Is SeS necessarily closed?

PROBLEM 3.8. Can the ideals in Corollary 1.11 be chosen to be closed?

PROBLEM 3.9. Let $n \in \mathbf{Z}^+$. Does the number of regular $\mathcal{J}$-classes of strongly π-regular subsemigroups of $\mathfrak{M}_n(K)$ have an upper bound (depending on n)? More generally, can $\mathcal{E}$ in Lemma 1.6 be replaced by a sufficiently large finite set of idempotents?

PROBLEM 3.10. Are the nil Rees factor semigroups of Theorem 1.10 and Corollary 1.11 necessarily nilpotent?

PROBLEM 3.11. Can the Krohn-Rhodes theorem for finite semigroups be generalized to strongly π-regular subsemigroups of $\mathfrak{M}_n(K)$?

EXAMPLE 3.12. Let $T_1 \subseteq K^m$, $T_2 \subseteq K^n$ be algebraic semigroups. Let $S = (T_1 \times \{0_n\} \times \{1\}) \cup (\{0_m\} \times T_2 \times \{0\}) \subseteq K^{m+n+1}$ where 0_m, 0_n are the zero vectors of

K^m and K^n respectively. Then S is closed. Define multiplication in S as follows.
$(a, b, \alpha)(c, d, \beta) = (\alpha\beta ac, (1-\alpha)\beta b + \alpha(1-\beta)d + (1-\alpha)(1-\beta)bd, \alpha\beta)$.
Then S is an algebraic semigroup. Let $\hat{T}_1 = T_1 \times \{0_n\} \times \{1\}$, $\hat{T}_2 = \{0_m\} \times T_2 \times \{0\}$. Then $S = \hat{T}_1 \cup \hat{T}_2$, $xy = yx = y$ for $x \in \hat{T}_1$, $y \in \hat{T}_2$. $\hat{T}_1, \hat{T}_2$ are disjoint closed subsemigroups of S. $\hat{T}_i$ is *-isomorphic to T_i $(i = 1, 2)$.

EXAMPLE 3.13. Let $\mathcal{A}$ be a finite dimensional algebra over K. Then the multiplicative semigroup of $\mathcal{A}$ is a connected algebraic semigroup. $\mathcal{A}$ along with the circle operation $a \circ b = a + b - ab$ is also a connected algebraic semigroup.

EXAMPLE 3.14. Let $S = \mathfrak{M}_n(K)$. For $i = 1, \ldots, n$, let $S_i = \{a \mid a \in S, \rho(a) < i\}$. If $e \in S_i$, $e^2 = e$, $\rho(e) = i$, then $SeS = S_i$ and so by Proposition 2.2, each S_i is a connected algebraic semigroup. S_1 is completely 0-simple and all subgroups of S_1 are abelian. Also $\dim S_1 = 2n - 1$.

Let S, T be algebraic semigroups. Suppose for $a \in S$, $b \in T$ an element $a^b \in S$ is uniquely determined. Suppose the map $(a, b) \to a^b$ is a morphism and that for all $a_1, a_2 \in S$, $b_1, b_2 \in T$, $(a_1a_2)^b = a_1^b a_2^b$, $(a_1)^{b_1b_2} = (a_1^{b_2})^{b_1}$. In $D = S \times T$ define $(a_1, b_1)(a_2, b_2) = (a_1a_2^{b_1}, b_1b_2)$. Then the *semidirect product* D is an algebraic semigroup. If S, T are connected then so is D. In particular if $a \in \mathfrak{M}_n(K)$, $b \in GL(n, K) = \{a \mid a \in \mathfrak{M}_n(K), \rho(a) = n\}$, we can set $a^b = bab^{-1}$. If G is any connected, closed subgroup of $GL(n, K)$, we can form the semidirect product of S_i (see Example 3.14) and G to again obtain a connected algebraic semigroup. By Lallement [**10**, Theorem 2.17], the semidirect product of S_1 and G is a subdirect product of completely simple and completely 0-simple semigroups.

EXAMPLE 3.15. The example $S = \{\left(\begin{smallmatrix}1 & a\\ 0 & b\end{smallmatrix}\right) \mid a, b \in K\}$ shows that Theorem 2.12 is not true without the assumption that the subgroups of S are abelian.

PROBLEM 3.16. Let S be a connected algebraic semigroup which is a semilattice of groups. Determine all possibilities for $E(S)$ and $|E(S)|$. For example, by Theorems 1.2 and 2.17, $|E(S)| \neq 3$. If S is also the multiplicative semigroup of a finite-dimensional algebra, then clearly $|E(S)| = 2^n$ for some $n \in \mathbf{Z}^+$. This is not true in general as the following example shows.

EXAMPLE 3.17. Let $T = K^4$ under multiplicaton and let $S = \{(a, b, c, d) \mid a, b, c, d \in K, ab = cd\}$. Then S is a connected, closed subsemigroup of T. S is also a semilattice of groups, $\dim S = 3$ and $|E(S)| = 10$.

PROBLEM 3.18. Determine all possibilities for $\Omega(S)$ and $|\Omega(S)|$ where S is a connected algebraic semigroup.

EXAMPLE 3.19. Let $T_1 = (K^3, *)$ where

$$(a_1, a_2, a_3) * (b_1, b_2, b_3) = (a_2b_3 + a_1 + b_1, b_2, a_3).$$

Let T_2 be any commutative finite-dimensional algebra with an identity element. Then T_1 is completely simple. T_1, T_2, $T_1 \times T_2$ are all examples of connected algebraic semigroups satisfying the hypothesis of Theorem 2.5.

EXAMPLE 3.20. Let $S = \{\left(\begin{smallmatrix}a & b\\ 0 & 0\end{smallmatrix}\right) \mid a, b \in K\}$. Then S is a connected algebraic semigroup of dimension 2. S is a semilattice of a nil semigroup and a right group.

EXAMPLE 3.21. Let $P \in \mathfrak{M}_n(K)$ and let $\mathcal{A} = \{A \mid A \in \mathfrak{M}_n(K), A^TPA = 0\}$, $\mathcal{B} = \{A \mid A \in \mathfrak{M}_n(K), A^TPA = P\}$. Then $\mathcal{A}$, $\mathcal{B}$ are closed subsemigroups of

$\mathfrak{M}_n(K)$. $\mathcal{A}$ has a zero and $\mathcal{B}$ has an identity element. When is $\mathcal{A}$ or $\mathcal{B}$ connected? If $n = 3$ and

$$P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then $\mathcal{A}$ is a connected algebraic semigroup of dimension 7.

References

1. G. Azumaya, *Strongly π-regular rings*, J. Fac. Sci. Hokkaido Univ. **13** (1954), 34–39.
2. A. Borel, *Linear algebraic groups*, Benjamin, New York, 1969.
3. W. E. Clark, *Remarks on the kernel of a matrix semigroup*, Czechoslovak Math. J. **15** **(90)** (1965), 305–310.
4. ______, *Affine semigroups over an arbitrary field*, Proc. Glasgow Math. Assoc. **7** (1965), 80–92.
5. ______, Private communication.
6. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. 1, Math. Surveys, No. 7, Amer. Math. Soc., Providence, R. I., 1961.
7. P. M. Cohn, *Universal algebra*, Harper and Row, New York, 1965.
8. M. Demazure and P. Gabriel, *Groupes algébriques*. Tome I: *Géométrie algébrique, généralités, groups commutatifs*, North-Holland, Amsterdam, 1970.
9. M. P. Drazin, *Pseudo-inverses in associative rings and semigroups*, Amer. Math. Monthly **7** (1958), 506–514.
10. G. Lallement, *Demi-groupes réguliers*, Ann. Mat. Pura Appl. **77** (1967), 47–129.
11. W. D. Munn, *Pseudo-inverses in semigroups*, Proc. Cambridge Philos. Soc. **57** (1961), 247–250.
12. M. Petrich, *Associative polynomial multiplications over an integral domain*, Math. Nachr. **29** (1965), 67–75.
13. R. J. Plemmons and R. Yoshida, *Generating polynomials for finite semigroups*, Math. Nachr. **47** (1970), 69–75.
14. M. S. Putcha, *Semilattice decompositions of semigroups*, Semigroup Forum **6** (1973), 12–34.
15. ______, *Quadratic semigroups on affine spaces*, Linear Algebra and Appl. **26** (1979), 107–121.
16. M. S. Putcha and J. Weissglass, *A semilattice decomposition into semigroups with at most one idempotent*, Pacific J. Math. **39** (1971), 225–228.
17. I. R. Shafarevich, *Basic algebraic geometry*, Springer-Verlag, Berlin and New York, 1974.
18. T. Tamura, *The theory of construction of finite semigroups*. I, Osaka J. Math. **8** (1956), 243–261.
19. ______, *Another proof of a theorem concerning the greatest semilattice decomposition of a semigroup*, Proc. Japan Acad. **40** (1964), 777–780.
20. T. Tamura and N. Kimura, *On decomposition of a commutative semigroup*, Kōdai Math. Sem. Rep. **4** (1954), 109–112.
21. R. Yoshida, *Algebraic systems which admit polynomial operations*, Mem. Res. Inst. Sci. Engrg. Ritsumeikan Univ. **10** (1963), 1–5.
22. ______, *On semigroups*, Bull. Amer. Math. Soc. **69** (1963), 369–371.

Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27650